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Exponentiated Cubic Transmuted Uniform Distribution: Properties and Applications

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ABSTRACT
In this article, we consider a generalized cubic transmuted distribution termed the Exponentiated Cubic Transmuted Uniform Distribution (ECTUD). We studied the mathematical properties as well as the reliability behaviour of the proposed distribution. The maximum likelihood estimation (MLE) method is used for the estimation of parameters. A simulation study is carried out to study the performance of the maximum likelihood estimates of the parameters. The practical applicability of the proposed distribution is illustrated using a real-world dataset.

KEYWORDS
exponentiated cubic transmuted distribution; moments; reliability analysis; order statistics; entropy; simulation.

1. Introduction

In modern statistics, generalizing probability distributions is a key technique that entails extending or modifying existing distributions to produce new families with more adaptability and better data-fitting capabilities. Real-world data complexity, such as skewness, heavy tails, and multimodality, is often difficult for classical distributions, making this approach crucial. For generalized distributions, these constraints can be overcome by adding one or more extra parameters, which allows greater control over features such as skewness, kurtosis, size, location, and tail behaviuor. There are several methods for generalization. Some of the most commonly used techniques for generalizing probability distributions include, the McDonald method proposed by McDonald [4], the Marshall–Olkin extended family proposed by Marshall and Olkin [3], the Exponentiated method introduced by Gupta et al.[5], the Beta-generated method proposed by Eugene et al.[6], the Quadratic rank transmutation map introduced by Shaw and Buckley[1], the Kumaraswamy-G families by Cordeiro and Castro[7], the Alpha power method by Mahdavi and Kundu [8] and the Cubic rank transmutation map developed by Granzotto et al.[2]. These approaches are widely applied in modern statistical mod-

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eling due to their ability to improve the flexibility of base distributions and enhance model fit across various types of data.

Transmutation is a statistical technique that is used to generate new probability distributions from the existing ones by introducing an additional transmutation parameter that adjusts the shape, scale, or other characteristics of the base model. The concept was first introduced by Shaw and Buckley [1] through the quadratic rank transmutation map (QRTM), which applies a rank-based transformation to the cumulative distribution function (cdf) of a baseline distribution. This approach permits improved flexibility in modeling real-world phenomena, particularly when baseline models fail to capture skewness, kurtosis, or tail behaviour. Since its introduction, the transmutation method has been extended to higher-order forms. The cubic rank transmutation method is an extension of the quadratic rank transmutation map designed to provide greater flexibility in the generation of new families of probability distributions. In this approach, the baseline cumulative distribution function is transformed through a cubic polynomial function allowing for even richer shapes and parameter control. In the literature, different forms of the cubic rank transmutation map are available. Al-Kadim and Mohammed [9] introduced the cubic transmuted Weibull distribution, Rahman et al. [10] proposed a new cubic transmuted uniform distribution.

The exponentiated method is one of the widely used generalization techniques applied to the existing probability models to enhance more flexibility. Given a baseline distribution with the cumulative distribution function $\psi(x)$, the exponentiated family is defined through the transformation,

$$F(x) = \left[\psi(x)\right]^{\theta}, \quad \theta > 0 \tag{1}$$

Consider the cubic rank transmutation map

$$\psi(x) = (1+\lambda)G(x) - 2\lambda G^2(x) + \lambda G^3(x); \quad |\lambda| \leq 1. \tag{2}$$

where $\psi(x)$ is the cumulative distribution function of the base distribution and G(x) is the cumulative distribution function of the cubic transmuted distribution.
The cumulative distribution function of the exponentiated cubic transmuted distribution is

$$F(x) = [(1+\lambda)G(x) - 2\lambda G^2(x) + \lambda G^3(x)]^{\theta}; \quad |\lambda| \leq 1. \tag{3}$$

Several distributions have been developed using the exponentiated method, including the exponentiated weibull distribution proposed by Pal et al. [11], the exponentiated exponential distribution introduced by Nadarajah [12], the exponentiated uniform distribution studied by Ramires et al. [14] and the exponentiated cubic transmuted weibull distribution proposed by Oseghale et al. [13]. In this article, we present a generalization of the cubic transmuted uniform distribution, namely the Exponentiated Cubic Transmuted Uniform Distribution (ECTUD), using the exponentiated technique.

2. Cubic Transmuted Uniform Distribution

The uniform distribution over the interval [0,1] has the probability density function(pdf) g(x) = 1 and the cumulative distribution function G(x) = x. Using the cdf of the standard uniform distribution in Eq. (2) we obtain the cumulative distribution function of the Cubic Transmuted Uniform Distribution (CTUD) as

$$\psi(x) = (1+\lambda)x - 2\lambda x^2 + \lambda x^3; \quad |\lambda| \leq 1, 0 \leq x \leq 1. \tag{4}$$

The of the CTUD is

$$\psi'(x) = (1+\lambda) - 4\lambda x + 3\lambda x^2; \quad |\lambda| \leq 1, 0 \leq x \leq 1. \tag{5}$$

3. Exponentiated Cubic Transmuted Uniform Distribution

In this section, we provide the formulation of the Exponentiated Cubic Transmuted Uniform Distribution (ECTUD) by applying the exponentiated technique to the cumulative distribution function of CTUD. When $\theta = 1$ the ECTUD will be reduced to CTUD. Using Eq. (4) in Eq. (1) the cumulative distribution function of the ECTUD is obtained as

$$F(x) = \left[(1+\lambda)x - 2\lambda x^2 + \lambda x^3\right]^{\theta}; 0 \leq x \leq 1,\ |\lambda| \leq 1, \theta > 0 \tag{6}$$

and the probability density function of the ECTUD is

$$f(x) = \theta\left[(1+\lambda) - 4\lambda x + 3\lambda x^2\right]\left[(1+\lambda)x - 2\lambda x^2 + \lambda x^3\right]^{\theta-1} \quad 0 \leq x \leq 1,\ |\lambda| \leq 1, \theta > 0. \tag{7}$$

3.1. Expansion for the pdf of ECTUD

From Eq. (7) consider the term

$$\left[(1+\lambda)x - 2\lambda x^2 + \lambda x^3\right]^{\theta-1} = x^{\theta-1}\left[(1+\lambda) - 2\lambda x + \lambda x^2\right]^{\theta-1}$$

Factor $(1+\lambda)$ to apply the binomial series expansion $\left[(1+\lambda) - 2\lambda x + \lambda x^2\right]^{\theta-1}$

$$\left[(1+\lambda) - 2\lambda x + \lambda x^2\right]^{\theta-1} = (1+\lambda)^{\theta-1}\left[1 + \frac{-2\lambda x + \lambda x^2}{1+\lambda}\right]^{\theta-1}$$

Apply the binomial series $(1+v)^{\theta-1} = \sum_{i=0}^{\infty}\binom{\theta-1}{i}v^i$

$$\left[(1+\lambda) - 2\lambda x + \lambda x^2\right]^{\theta-1} = (1+\lambda)^{\theta-1}\sum_{i=0}^{\infty}\binom{\theta-1}{i}\left(\frac{-2\lambda x + \lambda x^2}{1+\lambda}\right)^i.$$

Now expanding $(-2\lambda x + \lambda x^2)^i = \lambda^i(-2x + x^2)^i$

$$= \lambda^i \sum_{j=0}^{i} \binom{i}{j} (-2)^{i-j} x^{i+j}$$

Therefore,

$$\left[(1+\lambda)x - 2\lambda x^2 + \lambda x^3\right]^{\theta-1} = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \binom{\theta-1}{i} \binom{i}{j} (1+\lambda)^{\theta-1-i} \lambda^i (-2)^{i-j} \left[x\right]^{\theta+i+j-1}$$

Finally, the expanded pdf of the ECTUD becomes

$$f(x) = \theta(1+\lambda) W_{ij} x^{\theta+i+j} - 4\theta\lambda W_{ij} x^{\theta+i+j+1} + 3\theta\lambda W_{ij} x^{\theta+i+j+2}; 0 \le x \le 1. \quad (8)$$

where

$$W_{ij} = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \binom{\theta-1}{i} \binom{i}{j} (1+\lambda)^{\theta-i-1} \lambda^i (-2)^{i-j}.$$

The probability density function and cumulative distribution function plots of the ECTUD for different combinations of parameter values are displayed in Figure 1 and Figure 2 to illustrate the behaviour of the distribution.

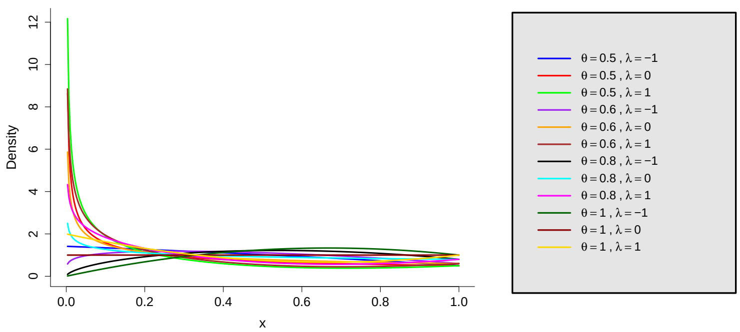

Figure 1. pdf of the ECTUD for different values of parameters.

The plots together highlight the influence of the parameters θ and λ on the distribution defined over the interval [0,1]. The pdf plot indicates that smaller values of θ yield sharp peaks near $x = 0$, particularly when $\lambda = -1$, reflecting a strong concentration of probability mass close to the origin, whereas larger values of θ flatten the density and distribute it more evenly across the support. The parameter λ establishes the skewness primarily. When $\lambda = 1$, the density is concentrated near the lower end of the interval, while $\lambda = -1$ shifts the probability mass towards higher values of x. The cumulative distribution function plots strengthen these findings, showing that for $\lambda = 1$, the cumulative distribution function increases steeply near the origin, indicating a rapid accumulation of probability for small x, while for $\lambda = -1$, the cumulative distribution function grows more slowly before increasing sharply, reflecting a heavier right tail. In

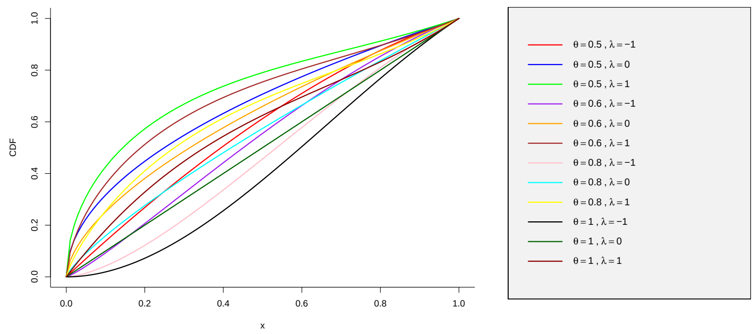

Figure 2. cdf of the ECTUD for different values of parameters.

general, θ governs the decay rate and spread of the distribution, while λ controls its skewness, thus providing a flexible family capable of modeling data concentrated near either boundary of the unit interval.

4. Distributional Properties of ECTUD

In this section, some mathematical properties such as moments, variance, moment generating function, and characteristic function of the ECTUD are derived.

Theorem:4.1
If X follows the ECTUD with parameters θ and λ, where $|\lambda| \leq 1$ and $\theta > 0$, then the rth moment of X is given as follows.

$$\mu'_r = \theta(1+\lambda)W_{ij}\frac{1}{\theta+i+j+r+1} - 4\theta\lambda W_{ij}\frac{1}{\theta+i+j+r+2} + 3\theta\lambda W_{ij}\frac{1}{\theta+i+j+r+3}.$$

Proof:
Let the random variable X have the ECTUD distribution, then the rth moment of X is given as

$$\mu'_r = \int_0^1 x^r\theta\left[(1+\lambda) - 4\lambda x + 3\lambda x^2\right]\left[(1+\lambda)x - 2\lambda x^2 + \lambda x^3\right]^{\theta-1}dx$$

using the expanded pdf in Eq. (8),

$$\mu'_r = \int_0^1 x^r\,\theta(1+\lambda)W_{ij}x^{\theta+i+j} - 4\theta\lambda W_{ij}x^{\theta+i+j+1} + 3\theta\lambda W_{ij}x^{\theta+i+j+2}dx$$

$$= \int_0^1 \theta(1+\lambda)W_{ij}x^{\theta+i+j+r} - 4\theta\lambda W_{ij}x^{\theta+i+j+r+1} + 3\theta\lambda W_{ij}x^{\theta+i+j+r+2}dx$$

$$W_{ij} = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{\theta-1}{i}\binom{i}{j}(1+\lambda)^{\theta-i-1}\lambda^{i}(-2)^{i-j}$$

Therefore, the rth moment of the ECTUD is obtained as

$$\mu_r' = \theta(1+\lambda)W_{ij}\frac{1}{\theta+i+j+r+1} - 4\theta\lambda W_{ij}\frac{1}{\theta+i+j+r+2} + 3\theta\lambda W_{ij}\frac{1}{\theta+i+j+r+3}. \quad (9)$$

4.1. Mean

Putting r=1 in Eq. (8), the arithmetic mean of ECTUD is

$$\mu_1' = \theta(1+\lambda)W_{ij}\frac{1}{\theta+i+j+2} - 4\theta\lambda W_{ij}\frac{1}{\theta+i+j+r+3} + 3\theta\lambda W_{ij}\frac{1}{\theta+i+j+r+4}.$$

4.2. Variance

The variance of the ECTUD is obtained as

$$\mu_2 = \left[\theta(1+\lambda)W_{ij}\frac{1}{\theta+i+j+3} - 4\theta\lambda W_{ij}\frac{1}{\theta+i+j+r+4} + 3\theta\lambda W_{ij}\frac{1}{\theta+i+j+r+5}\right] - \Bigg(\theta(1+\lambda)W_{ij}\frac{1}{\theta+i+j+2} - 4\theta\lambda W_{ij}\frac{1}{\theta+i+j+r+3} + 3\theta\lambda W_{ij}\frac{1}{\theta+i+j+r+4})^2. \quad (10)$$

and the standard deviation is

$$SD = \Bigg[\theta(1+\lambda)W_{ij}\frac{1}{\theta+i+j+3} - 4\theta\lambda W_{ij}\frac{1}{\theta+i+j+r+4} + 3\theta\lambda W_{ij}\frac{1}{\theta+i+j+r+5} - \left(\theta(1+\lambda)W_{ij}\frac{1}{\theta+i+j+2} - 4\theta\lambda W_{ij}\frac{1}{\theta+i+j+r+3} + 3\theta\lambda W_{ij}\frac{1}{\theta+i+j+r+4}\right)^2\Bigg]^{1/2}. \quad (11)$$

4.3. Skewness and Kurtosis

The coefficient of variation, the skewness and kurtosis measures can now be calculated using the following relationships

$$CV(X) = \frac{\sqrt{\mathrm{Var}(X)}}{E(X)}.$$

$$\text{Skewness}(X) = \frac{E\left[(X-E(X))^3\right]^2}{[\mathrm{Var}(X)]^3}.$$

$$\text{Kurtosis}(X) = \frac{E\left[(X - E(X))^4\right]}{\left[\text{Var}(X)\right]^2}.$$

Table 1 represents the mean, variance, coefficient of variation, skewness, and kurtosis for different combinations of the parameter values of the ECTUD. The results show

Table 1. Mean, Variance, CV and Kurtosis of ECTUD

θ	λ	Mean	Variance	CV	Skewness	Kurtosis
2	-0.5	0.550000	0.135595	0.669514	-0.540989	1.704464
2	0.5	0.450000	0.159405	0.887234	-0.043210	1.266111
2	0.8	0.420000	0.162648	0.960228	0.101994	1.251828
2	1.0	0.400000	0.163810	1.011835	0.199672	1.269200
5	-0.5	0.773810	0.074830	0.353512	-2.007610	0.656109
5	0.5	0.654762	0.147676	0.586910	-0.980284	2.198760
5	0.8	0.619048	0.164002	0.654185	-0.766347	1.767366
5	1.0	0.595238	0.173469	0.699714	-0.635537	1.556590
10	-0.5	0.883700	0.032262	0.203254	-3.909790	19.270513
10	0.5	0.782967	0.112237	0.427883	-1.809159	4.505466
10	0.8	0.752747	0.132273	0.483154	-1.514538	3.449513
10	1.0	0.732601	0.144615	0.519086	-1.347765	2.945033

that increasing θ shifts the distribution to larger values while also making it more concentrated and stable, as indicated by decreasing the variance and coefficient of variation. In contrast, increasing λ generally reduces the mean and increases the spread of the distribution. The distribution is predominantly left-skewed, and this skewness becomes stronger as θ increases, reflecting a higher concentration of probability near higher values with a longer tail on the left. Kurtosis patterns indicate that for smaller θ the distribution tends to have lighter tails, whereas for larger θ it becomes more peaked and heavy-tailed, especially for certain values λ. In general, θ mainly controls the location and sharpness of the distribution, while λ influences its spread and tail behaviour. Figure 3 represents the changes in the skewness and kurtosis of the ECTUD with different combinations of the parameter values.

Figure 3. Skewness and kurtosis plots of the ECTUD for different parameter values

Theorem:4.2 If X follows the ECTUD with parameters θ and λ, where $|\lambda| \leq 1$

and $\theta > 0$, then the moment generating function of x is given as

$$M_x(t) = \theta \sum_{r=0}^{\infty} \frac{t^r}{r!}[(1+\lambda)W_{ij}\frac{1}{\theta+i+j+r+1} - 4\lambda W_{ij}\frac{1}{\theta+i+j+r+2} + 3\lambda W_{ij}\frac{1}{\theta+i+j+r+3}].$$

Proof:
Let X has the ECTUD distribution, then the moment generating function of X is given as

$$M_x(t) = \int_0^1 e^{tx}\,\theta\left[(1+\lambda) - 4\lambda x + 3\lambda x^2\right]\left[(1+\lambda)x - 2\lambda x^2 + \lambda x^3\right]^{\theta-1}\,dx$$

$$= \int_0^1 e^{tx}\,\theta(1+\lambda)W_{ij}x^{\theta+i+j} - 4\theta\lambda W_{ij}x^{\theta+i+j+1} + 3\theta\lambda W_{ij}x^{\theta+i+j+2}dx$$

using the exponential series expansion,

$$M_x(t) = \int_0^1 \sum_{r=0}^{\infty} \frac{(tx)^r}{r!}\,f(x)\,dx$$

since $x, t > 0$ and $f(x) > 0$, applying Tonelli's theorem,

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\int_0^1 x^r f(x)dx$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}\,\mu_r'$$

Therefore, we obtain the mgf of the ECTUD as

$$M_x(t) = \theta \sum_{r=0}^{\infty} \frac{t^r}{r!}[(1+\lambda)W_{ij}\frac{1}{\theta+i+j+r+1} - 4\lambda W_{ij}\frac{1}{\theta+i+j+r+2} + 3\lambda W_{ij}\frac{1}{\theta+i+j+r+3}]. \quad (12)$$

5. Order Statistics

Let $X_1, X_2, \ldots, X_n$ be the random sample taken from the ECTUD with the probability density function $g_X(x)$ and the cumulative distribution function $G_X(x)$. Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be its order statistics. The probability density function of the k-th order statistic $X_{(k)}$ is given by,

$$g_{X_{(k)}}(x) = \frac{n!}{(k-1)!\,(n-k)!}\,g_X(x)\left[G_X(x)\right]^{k-1}\left[1 - G_X(x)\right]^{n-k}. \quad (13)$$

Using Eq. (6) and Eq. (7)

$$g_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!}\,\theta\left[(1+\lambda)-4\lambda x+3\lambda x^2\right]\left[(1+\lambda)x-2\lambda x^2+\lambda x^3\right]^{\theta k-1}$$
$$\times\left[1-\left((1+\lambda)x-2\lambda x^2+\lambda x^3\right)^{\theta}\right]^{n-k}$$

Therefore, the probability density function of the higher-order statistic $X_{(n)}$ of the ECTUD can be obtained as

$$g_{X_{(n)}}(x) = n\theta\big[(1+\lambda)-4\lambda x+3\lambda x^2\big]\left[(1+\lambda)x-2\lambda x^2+\lambda x^3\right]^{n\theta-1}. \tag{14}$$

The probability density function of the first-order statistic $X_{(1)}$can be obtained as

$$g_{X_{(1)}}(x) = n\theta\big[(1+\lambda)-4\lambda x+3\lambda x^2\big]\left[(1+\lambda)x-2\lambda x^2+\lambda x^3\right]^{\theta-1}$$
$$\times\left[1-\big((1+\lambda)x-2\lambda x^2+\lambda x^3\big)\right]^{\theta(n-1)}. \tag{15}$$

6. Renyi Entropy

If a random variable X has the ECTUD distribution, then the Renyi entropy of X is

$$R_\alpha(x) = \frac{1}{1-\alpha}\log\int_0^1 f^\alpha(x)\,dx.$$

$$= \frac{\theta^\alpha}{1-\alpha}\log\int_0^1\left[(1+\lambda)-4\lambda x+3\lambda x^2\right]^\alpha\left[(1+\lambda)x-2\lambda x^2+\lambda x^3\right]^{\alpha(\theta-1)}dx$$

$$= \frac{1}{1-\alpha}\log\left[\theta^\alpha\int_0^1\left[1+\lambda(1-4x+3x^2)\right]^\alpha\left((1+\lambda)x\Big(1+\frac{\lambda x^2-2\lambda x}{1+\lambda}\Big)\right)^{\alpha(\theta-1)}dx\right]$$

$$= \frac{1}{1-\alpha}\log\left[\theta^\alpha\int_0^1\left[1+\lambda(1-4x+3x^2)\right]^\alpha\left((1+\lambda)x\Big(1+\frac{\lambda x^2-2\lambda x}{1+\lambda}\Big)\right)^{\alpha(\theta-1)}dx\right]$$

by series expansion,

$$R_\alpha = \frac{1}{1-\alpha}\log\left[\theta^\alpha\int_0^1\sum_{j=0}^{\infty}\binom{\alpha}{j}\left[\lambda(1-4x+3x^2)\right]^j(1+\lambda)^{\alpha(\theta-1)}x^{\alpha(\theta-1)}\sum_{k=0}^{\infty}\binom{\alpha(\theta-1)}{k}\left(\frac{\lambda x^2-2\lambda x}{1+\lambda}\right)^k dx\right].$$

Therefore, the Renyi entropy of the ECTUD distribution is

$$R_\alpha = \frac{1}{1-\alpha}\log\left[\theta^\alpha(1+\lambda)^{\alpha(\theta-1)}\sum_{j=0}^{\infty}\binom{\alpha}{j}\lambda^j\sum_{k=0}^{\infty}\binom{\alpha(\theta-1)}{k}\left(\frac{\lambda}{1+\lambda}\right)^k I^*\right]. \tag{16}$$

$$I^* = \int_0^1 x^{\alpha(\theta-1)+k} \left(1 - 4x + 3x^2\right)^j (x-2)^k \, dx.$$

7. Reliability Analysis

In this section, we present the survival function, hazard rate function, cumulative hazard rate function, reverse hazard rate function and odds function for the proposed distribution.

7.1. Survival Function

For a random variable X representing the time until an event occurs, the survival function is defined as $R(x) = P(X > x)$
The survival function of the ECTUD random variable X is obtained as

$$R(x) = 1 - \left[(1+\lambda)x - 2\lambda x^2 + \lambda x^3\right]^\theta ; 0 \leq x \leq 1. \tag{17}$$

7.2. Hazard Rate Function

The hazard rate function is a key concept in reliability theory and survival analysis which represents the instantaneous rate at which events occur, conditional on their non-occurrence up to that time.
For a continuous random variable X, the hazard rate function is defined as

$$h(x) = \frac{f(x)}{S(x)}$$

The hazard rate function of the ECTUD random variable X is obtained as

$$h(x) = \frac{\theta\left((1+\lambda) - 4\lambda x + 3\lambda x^2\right)\left((1+\lambda)x - 2\lambda x^2 + \lambda x^3\right)^{\theta-1}}{1 - \left((1+\lambda)x - 2\lambda x^2 + \lambda x^3\right)^\theta} ; 0 \leq x \leq 1. \tag{18}$$

The limiting behaviour of the hazard rate function is given below:
Let

$$h(x) = \frac{\theta\, A(x)\, B(x)^{\theta-1}}{1 - B(x)^\theta}$$

where $A(x) = (1+\lambda) - 4\lambda x + 3\lambda x^2$ and $B(x) = (1+\lambda)x - 2\lambda x^2 + \lambda x^3$
So, $A(1) = 1$ and $B(1) = 1$
The first order Taylor series expansion of B(x) around x=1 is

$$B(x) \approx B(1) + A(1)(x-1)$$

$$\approx 1 - (1-x)$$

Thus, as $x \longrightarrow 1^-$

$$\theta A(x)B(x)^{\theta-1} \longrightarrow \theta$$

and using the binomial approximation,

$$B(x)^\theta = (1-(1-x))^\theta \approx 1-\theta(1-x)$$

then

$$1 - B(x)^\theta \;\approx\; \theta(1-x)$$

$$h(x) = \frac{\theta}{\theta(1-x)}$$

Therefore,

$$\lim_{x \to 1^-} h(x) = +\infty.$$

Since ECTUD has bounded support in [0,1], the hazard rate tends to infinity, as x tends to 1. This indicates that the failure rate of the system becomes extremely high as its lifetime approaches the maximum bound.

7.3. Cumulative Hazard Rate Function

The cumulative hazard rate is a measure of the total amount of risk that has been accumulated up to a certain time t. It gives the overall risk of the event occurring by time t, based on the hazard rate function h(x).

$$H(x) = \int_0^t h(u)\,du$$

There is a direct connection between the cumulative hazard function and the survival function.

$$R(x) = e^{-H(x)}$$

The cumulative hazard rate function of the ECTUD is

$$H(x) = -\ln R(x) = -\ln\left[1 - \left((1+\lambda)x - 2\lambda x^2 + \lambda x^3\right)^\theta\right]; 0 \le x \le 1. \tag{19}$$

7.4. Reverse Hazard Rate Function

The reverse hazard rate represents the instantaneous risk of failure in the remaining time interval, conditional on the event having already occurred at time t.

If f(x) is the probability density function and F(x) is the cumulative distribution function of the ECTUD random variable X, then

$$r(x) = \frac{f(x)}{F(x)} = \frac{\theta\left((1+\lambda) - 4\lambda x + 3\lambda x^2\right)}{(1+\lambda)x - 2\lambda x^2 + \lambda x^3}; 0 \le x \le 1. \tag{20}$$

7.5. Odds function

The odds function in survival analysis is another way to express the risk of a system related to the hazard rate and survival function.
The odds function of the ECTUD random variable X is obtained as

$$O(x) \;=\; \frac{\left((1+\lambda)x - 2\lambda x^2 + \lambda x^3\right)^{\theta}}{1 - \left((1+\lambda)x - 2\lambda x^2 + \lambda x^3\right)^{\theta}}; 0 \le x \le 1. \tag{21}$$

Figure 4, Figure 5 and Figure 6 represent the survival function, the hazard rate function, and the reverse hazard rate function of the ECTUD for different combinations of parameters θ and λ. The reliability function consistently declines with increasing x, showing the expected reduction in the reliability of the system over time. However, the rate of decline is strongly parameter dependent. Smaller values of θ or negative λ correspond to shorter system lifetimes, while larger values of θ or positive λ slow down the decay, indicating that the system remains reliable over a longer range of x. The hazard curves in the figure describe that the model provides substantial shape flexibility driven by the interaction of θ and λ, producing increasing, decreasing, or bathtub-like behaviours in the early and middle ranges. As x increases, λ adjusts the curvature and skewness, further enhancing the adaptability of the model. Regardless of the parameter combination, all curves eventually rise sharply near the upper bound, indicating an accelerating failure tendency.

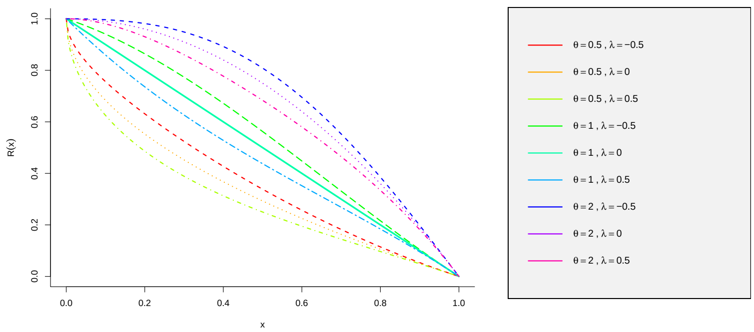

Figure 4. Survival function of the ECTUD for different values of parameters

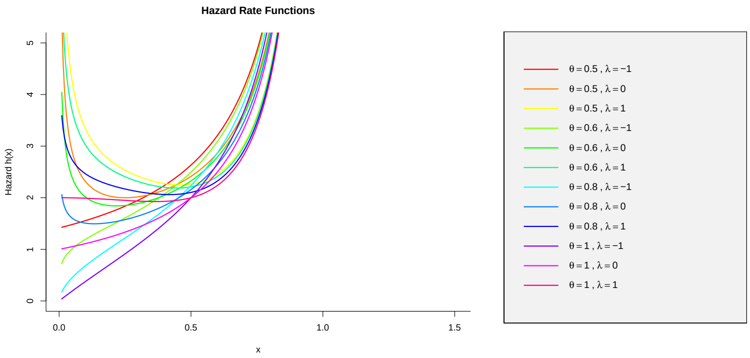

Figure 5. Hazard rate function of the ECTUD for different values of parameters

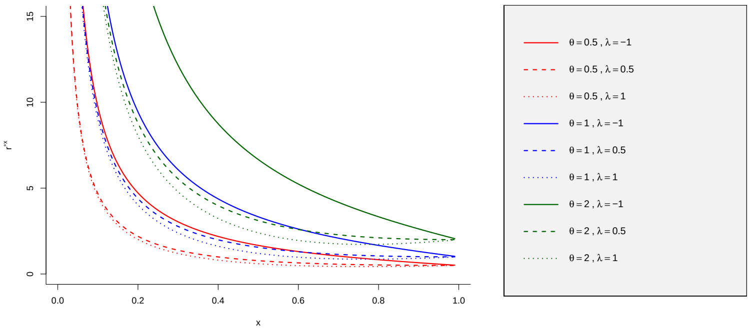

Figure 6. Reverse Hazard rate function of the ECTUD for different values of parameters

8. Estimation of Parameters

In this section, we discuss the estimation of the parameters θ and λ of the ECTUD using the method of maximum likelihood estimation.

$$L = \prod_{i=1}^{n} \theta\left[(1+\lambda) - 4\lambda x_i + 3\lambda x_i^2\right]\left[(1+\lambda)x_i - 2\lambda x_i^2 + \lambda x_i^3\right]^{\theta-1}$$

$$\log L = n\log\theta + \sum_{i=1}^{n}\log\left[(1+\lambda) - 4\lambda x_i + 3\lambda x_i^2\right] + (\theta-1)\sum_{i=1}^{n}\log\left[(1+\lambda)x_i - 2\lambda x_i^2 + \lambda x_i^3\right]$$

differentiating partially,

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\log\left[(1+\lambda)x_i - 2\lambda x_i^2 + \lambda x_i^3\right] = 0$$

$$\hat{\theta} = -\frac{n}{\sum_{i=1}^{n} \log\Big((1+\hat{\lambda})x_i - 2\hat{\lambda}x_i^2 + \hat{\lambda}x_i^3\Big)}. \tag{22}$$

$$\frac{\partial \log L}{\partial \lambda} = \sum_{i=1}^{n} \frac{1-4x_i+3x_i^2}{1+\lambda\big(1-4x_i+3x_i^2\big)} + (\theta-1)\sum_{i=1}^{n} \frac{1-2x_i+x_i^2}{1+\lambda\big(1-2x_i+x_i^2\big)} = 0. \tag{23}$$

By Eq. (22) and Eq. (23) we get the estimates of θ and λ. Eq. (23) does not have a closed form. Due to the complex structure of these equations, we employ R software to solve them numerically.

9. Simulation

This section evaluates the performance of the MLEs for the parameters θ and λ of the ECTUD distribution using the Monte Carlo simulation. Samples were generated with sizes n = 25, 50, 75, 100, 200, and 500 for fixed parameter values. Random numbers are generated from the ECTUD using the inverse transform method. Equating the cumulative distribution function of the ECTUD with a uniform random number and inverting the expression. That is

$$\big[(1+\lambda)x - 2\lambda x^2 + \lambda x^3\big]^{\theta} - u = 0$$

where $u \sim U(0,1)$ solved numerically for x. A bracketing-based root-finding procedure is employed to determine the solution within a prescribed interval using the R software. This ensured a stable and accurate numerical inversion of the cumulative distribution function, thereby producing reliable random samples from the ECTUD. The estimates of the unknown parameters were obtained by maximizing the total log-likelihood function using the L-BFGS-B algorithm . The estimated values of the parameters with their corresponding Mean, Standard Error, Bias, and MSE for $\theta = 1$ and $\lambda = 0.5$ are given in Table 2 and for $\theta = 3$ and $\lambda = 1$ are in Table 3.

The Monte Carlo simulation results presented in Table 2 and Table 3 evaluate the performance of the Maximum Likelihood Estimator (MLE) for the parameters of the ECTUD distribution under two parameter settings, namely $\theta = 1$, $\lambda = 0.5$ and $\theta = 3$, $\lambda = 1$. In both scenarios, the MLE exhibits the property of consistency, with the mean estimates converging toward their true parameter values as the sample size increases. For small samples, the MLE of both parameters demonstrates relatively large biases and high mean squared errors, the bias being more pronounced for the estimation of λ, which also shows greater variability. However, as the sample size increases, bias, standard error, and MSE systematically decrease, confirming the improved efficiency and reliability of MLE in larger samples. For parameter setting $\theta = 1$, $\lambda = 0.5$, the MLE converges more rapidly to the true values, and when $\theta = 3$, $\lambda = 1$, the initial estimates, particularly for λ, deviate considerably from the true values, but the accuracy improves substantially with increasing sample size. By n=500, the MLEs of both θ and λin both cases are very close to the true values of the parameters, with minimal bias, low variability, and reduced MSE. Overall, the findings confirm that although the MLE may yield unstable estimates in small samples, it demonstrates

Table 2. Mean, Standard Error, Bias and MSE of the ECTUD distribution for $\theta = 1$and $\lambda = 0.5$

n	Parameter	Mean	SE	Bias	MSE
25	θ	0.9966917	0.29458707	-0.003308301	0.086618924
	λ	0.3519083	0.70833193	-0.148091707	0.522661804
50	θ	0.9879740	0.23951340	-0.012026018	0.057396560
	λ	0.4181936	0.65270065	-0.081806390	0.431858383
75	θ	1.0065890	0.19883530	0.006589884	0.039499822
	λ	0.4603421	0.53930239	-0.039657853	0.291838116
100	θ	1.0042652	0.19901731	0.004265218	0.039546865
	λ	0.4425634	0.54948594	-0.057436625	0.304629898
200	θ	0.9902620	0.14438949	-0.009738029	0.020901456
	λ	0.4413868	0.43115785	-0.058613197	0.188960802
500	θ	1.0056505	0.09043434	0.005650501	0.008193941
	λ	0.5047794	0.26200099	0.004779374	0.068530073

Table 3. Mean, Standard Error, Bias and MSE of the ECTUD distribution for $\theta = 3$and $\lambda = 1$

n	Parameter	Mean	SE	Bias	MSE
25	θ	2.6412843	0.7509912	-0.35871573	0.69153675
	λ	0.4218835	0.7585551	-0.57811648	0.90843764
50	θ	2.6694549	0.6274674	-0.33054511	0.50218793
	λ	0.5641691	0.6376013	-0.43583089	0.59567089
75	θ	2.7989185	0.5001368	-0.20108148	0.29007033
	λ	0.7192432	0.4551561	-0.28075679	0.28557708
100	θ	2.8411163	0.4180094	-0.15888365	0.19962637
	λ	0.7666424	0.3904979	-0.23335760	0.20663938
200	θ	2.8581174	0.3055811	-0.14188256	0.11332370
	λ	0.8309392	0.2516647	-0.16906077	0.09179900
500	θ	2.9173320	0.1806272	-0.08266797	0.03934993
	λ	0.9075512	0.1423684	-0.09244883	0.02877500

strong asymptotic properties, providing accurate and efficient parameter estimation for the ECTUD distribution in sufficiently large samples.

10. Data Analysis

In this section, we provide the data analysis in order to assess the goodness of fit of the ECTUD. Table 4 presents the dataset used by Rahman et al.[10] on the lifetimes of 30 electronic devices measured in days, which are rescaled data to ensure compatibility with the support of the proposed distribution. Since the model is defined on the unit interval [0,1], the data is normalized using $x_i = \frac{t_i - \min(t)}{\max(t) - \min(t)}$. This normalization preserves the ordering and relative dispersion of the lifetimes while aligning the data with the theoretical domain of the proposed distribution. The ECTUD is fitted to the subject data and compared with the Transmuted Weibull Distribution (TWD), the Cubic Transmuted Weibull distribution(CTWD) and the Exponentiated Cubic Transmuted Weibull Distribution (ECTWD). Estimates of all model parameters are computed using the maximum likelihood technique. Maximum lilelihood estimates of the distribution parameters, along with the standard errors, are shown in Table 5. The criteria used to discriminate the models are the AIC and the BIC. A distribution is said to provide the best fit to the data if, among all the distributions under consideration, it corresponds to minimum values of AIC and BIC. We also provide the Kolmogorov-Smirnov test (K-S test) to check the goodness-of-fit of the distribution with the considered data set. Table 6 represents the model comparison using the criteria mentioned above. Based on the results presented in Table 6, it is evident

Table 4. Lifetimes of 30 Electronic Devices

0.020	0.029	0.034	0.044	0.057	0.096	0.106	0.139	0.156	0.164
0.167	0.177	0.250	0.326	0.406	0.607	0.650	0.672	0.676	0.736
0.817	0.838	0.910	0.931	0.946	0.953	0.961	0.981	0.982	0.990

Table 5. Parameter estimates and standard errors for selected distributions

Distribution	Parameters	SE
TWD	$\alpha = 1.0991,\ \beta = 0.4849,\ \lambda = -0.1231$	$\alpha = 0.1995,\ \beta = 0.1262,\ \lambda = 0.4168$
CTWD	$\alpha = 2.2626,\ \beta = 1.304,\ \lambda = 0.7333$	$\alpha = 0.4947,\ \beta = 0.2224,\ \lambda = 0.6404$
ECTWD	$\theta = 0.0801,\ \alpha = 0.8609,\ \beta = 8.6392,\ \lambda = -0.8748$	$\theta = 0.0148,\ \alpha = 0.02889,\ \beta = 0.2305,\ \lambda = 0.2190$
ECTUD	$\theta = 1.0762,\ \lambda = 0.99995$	$\theta = 0.1965,\ \lambda = 0.00609$

Table 6. Model selection criteria for candidate distributions

Distribution	log likelihood	AIC	BIC	AICc	KS	P value
TWD	-8.5275	23.055	27.2586	23.9781	0.2932	0.0088
CTWD	-7.909	21.818	26.0216	22.7411	0.3202	0.0031
ECTWD	-37.5441	83.0883	88.6930	84.6883	0.3581	0.0006
ECTUD	2.7545	-1.509	1.2934	-1.0646	0.1820	0.2421

that the proposed ECTUD provides a superior fit compared to the other distributions considered.

11. Conclusion

In this paper, we provide a generalization of the Cubic Transmuted Uniform Distribution, namely the Exponentiated Cubic Transmuted Uniform Distribution(ECTUD). This generalization is constructed by applying the exponentiated technique to the standard model, thereby introducing an additional parameter that enhances the flexibility of the model in capturing diverse data behaviours. The proposed distribution is thoroughly examined to establish its theoretical properties. We derive explicit analytical expressions for its probability density function and cumulative distribution function, enabling a deeper understanding of its behaviour under varying parameter values. Furthermore, we explore several important statistical characteristics of the model, including moments, mean, variance, order statistics, and Renyi entropy, to assess its descriptive power in diverse contexts. The limiting behaviour of the hazard rate function reflects the characteristics of realistic lifetime processes with finite maximum lifespan, where the probability of survival becomes extremely small as its lifetime approaches maximum bound. For parameter estimation, we employ the maximum likelihood method and a simulation study is carried out to see the performance of the estimates. The proposed distribution offers substantial flexibility for modeling lifetime and reliability data. Comparative analysis with other established distributions using a real-world dataset showed that the ECTUD provides a markedly superior fit. The findings in this study demonstrate that the proposed distribution can serve as an effective extension, capable of capturing a broader range of empirical phenomena and offering improved performance in practical applications.

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